12

Answers and Hints

12.3 Haggling at the Bazaar

(c) Let $\mu = \mathrm{P}(b_h|\text{refuse})$. Then by Bayes' Rule, with $x = P(refuse|b_h)$,

$$\begin{aligned}\mu &= \frac{\mathrm{P}(\text{refuse}|b_h)\mathrm{P}(b_h)}{\mathrm{P}(\text{refuse}|b_h)\mathrm{P}(b_h) + \mathrm{P}(\text{refuse}|b_l)\mathrm{P}(b_l)} \\ &= \frac{x\pi}{x\pi + 1 - \pi} \\ &= \frac{\pi}{\pi + \frac{1-\pi}{x}} \\ &\leq \pi.\end{aligned}$$

(f) All buyers accept $p_1 = b_l$, so $p_1 < b_l$ is dominated. Any player who accepts p_1 for $b_l < p_1 < p^*$ accepts $p_1 = p^*$. No buyer accepts $p_1 > p^*$, since both high and low types prefer to wait till the second round, and get $p_2 = b_l$. At $p_1 = p^*$, the payoffs to "accept" and "reject" are equal for a high-value buyer, since then $b_h - p_1 = \delta_b(b_h - b_l)$, so such a buyer accepts on round one.

(i) Suppose the seller chooses $p_2 = b_l$. Since the seller's posterior probability for $\{b = b_l\}$ cannot be less than π (for the same reason as in the last problem), and since he would charge b_h in the one-shot game, he must charge $p_2 = b_h$. So suppose the seller chooses $p_2 = b_h$. Then the only undominated strategies on the first round are $p_1 = \delta_b b_h$ and $p_1 = b_l$. But if a buyer rejects $p_1 = \delta_b b_h$, he must be a low-value buyer, so it is not subgame perfect to charge $p_2 = b_h$.

12.8 A Common Value Auction: The Winner's Curse

If $\tilde{v}$ is the measured value for any player, then

$$\mathrm{P}[\tilde{v} \leq x|v] = \frac{x - v + a}{2a}.$$

VISIT…

Suppose your measured value, and hence bid, is x. The probability $F(x)$ that yours is the highest bid is just the probability that all $n-1$ other measured values are less than x, which is given by

$$F(x) = \left(\frac{x - v + a}{2a}\right)^{n-1}.$$

Your expected profit π is given by

$$\begin{aligned}\pi &= \frac{1}{2a}\int_{v-a}^{v+a}(v-x)F(x)dx \\ &= \frac{1}{(2a)^n}\int_{v-a}^{v+a}(v-x)(x-v+a)^{n-1}dx \\ &= -a\frac{n-1}{n(n+1)}\end{aligned}$$

12.9 Common Value Auction II

(a) If GE bids b, it wins if $b > (\alpha+\beta)y/2\beta$, or $2b\beta/(\alpha+\beta) > y$. GE's expected profit is thus

$$\int_0^{\frac{2b\beta}{\alpha+\beta}}(x+y-b)\frac{1}{\beta}dy = \frac{2b}{(\alpha+\beta)^2}\left[(\alpha+\beta)x - b\alpha\right].$$

If we equate the derivative of this expression with respect to b to zero, and solve for b, we get $b = x(\alpha+\beta)/2\alpha$. A similar reasoning holds with respect to Martin Marietta. GE's expected profit is then $x^2/2\alpha$, which can be found by substitution. GE then wins if $x/\alpha > y/\beta$, which clearly has probability 1/2.

(b) Each firm bids 1/2 of its known value, plus 1/2 of a random guess of the value it does not know. Notice that when α/β is very small, so kramel inhibitors are worth little at best, and GE knows the value of $\tilde{x}$, but the knowledge is practically worthless, GE's bid is roughly 1/2 of a random sample from the uniform distribution on $[0,\beta]$, and its expected profit is close to zero, though its probability of winning is still 1/2, which is still uniformly distributed on [0,1]. In the same situation, Martin Marietta bids one half of the actual value of the beta-phase detector and can expect big profits.

(c) If GE bids b having seen x, it wins if $b/\gamma > y$, so its expected profit is

$$\int_0^{\frac{b}{\gamma}} (x + y - b)\frac{1}{\beta}dy.$$

Evaluating this integral and setting its derivative to zero gives $b = \gamma x/(2\gamma - 1)$. Similarly for Martin Marietta.

(d) Suppose the value is $\tilde{x} + \tilde{y} + \tilde{z}$ where $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are uniformly distributed on $[0, \alpha]$, $[0, \beta]$, and $[0, \gamma]$, respectively. Suppose the three bidders bid νx, ηy, and ζz, respectively. Bidder 1 then must determine ν given η and ζ. The expected profit of bidder 1 who sees x and bids b is given by

$$\int_0^{\frac{b}{\eta}} \int_0^{\frac{b}{\zeta}} (x + y + x - b)\frac{dy}{\eta}\frac{dz}{\zeta}.$$

Evaluating this integral, setting its derivative with respect to b equal to zero, and solving for b, we get

$$b = F(\eta, \zeta)x,$$

where

$$F(\lambda, \mu) = \frac{4\lambda\mu}{3(2\lambda\mu - \lambda - \mu)},$$

so

$$\nu = F(\eta, \zeta). \tag{12.1}$$

If we substitute $\eta = \zeta = 5/3$, we see that $\nu^* = 5/3$, which proves the existence part of the assertion. For uniqueness, let (ν^*, η^*, ζ^*) be any solution. Note that if we repeat the above analysis to find η and ζ, we will find

$$\nu^* = F(\eta^*, \zeta^*)j, \eta^* = F(\zeta^*, \nu^*), \zeta^* = F(\eta^*, \nu^*).$$

If we substitute the first equation into the second, and solve for η^*, we get

$$\eta^* = \frac{7\zeta^*}{6\zeta^* - 3}.$$

Now substituting ν^* and $\eta*$ in the equation for ζ^*, we find $\zeta^* = 5/3$, so $\eta^* = 5/3$, and finally $\nu^* = 5/3$.

12.10 Predatory Pricing: Pooling and Separating Equilibria

If the equilibrium is separating, then the sane incumbent accommodates and earns $d_1(1+\delta)$. If the incumbent preyed in the first period, his earning would be $-p_1 + \delta m$. Thus a separating equilibrium requires (12.7). Clearly this condition is also sufficient, when supplemented by the entrant's belief that the incumbent preys only if crazy, which is of course true if the equilibrium is separating.

If there is a pooling equilibrium, clearly (12.7) must be violated, or else the sane incumbent would prefer to accommodate. In addition, the sane firm must induce exit, since otherwise it would not be profitable to prey in the first period. Thus we must have $\pi_2 d_2 - (1-\pi_2)p_2 \leq 0$, which becomes

$$\pi_2 \leq \frac{p_2}{d_2 + p_2}$$

where π_2 is the entrant's posterior on the incumbent being sane if he preys in the first period. Clearly $\pi_1 = \pi_2$ in this case, since all firms prey in the first period, and hence no information is gained by observing the behavior of the incumbent. Thus the condition for a pooling equilibrium is (12.8). Notice that if the frequency of sane firms is sufficiently high (i.e., if π_1 is sufficiently large), the pooling equilibrium cannot occur even if the incumbent gains from driving an entrant out of the market.

Let us show that if equations (12.7) and (12.8) are both violated, and if the entrant would enter the market if there were only one period, then we have a hybrid perfect Bayesian equilibrium, in which both parties randomize.

Let α be the probability that a sane incumbent preys in the first period, and let β be the probability that the entrant stays in the second period if the incumbent preys in the first period. The payoff to 'stay' in the second period if the incumbent preys in the first is

$$-(1-\pi_2)p_2 + \pi_2 d_2,$$

and this must equal the payoff to 'drop', which is zero. Thus we must have

$$\pi_2 = \frac{p_2}{d_2 + p_2}.$$

This is *prima facie* possible, since (12.8) is violated and $\pi_2 \leq \pi_1$.

To find β, note that the payoff to 'prey' is

$$\beta(-p_1 + \delta d_1) + (1-\beta)(-p_1 + \delta m),$$

and the payoff to 'accommodate' is $d_1(1+\delta)$. These must be equal, which gives (12.10). Note that β is always positive, and $\beta < 1$ precisely when the separating equilibrium condition (12.7) is violated.

But how do we get α? Obviously we must use Bayes' Rule. We have

$$\mathrm{P}[\text{sane}|\text{prey}] = \frac{\mathrm{P}[\text{prey}|\text{sane}]\mathrm{P}[\text{sane}]}{\mathrm{P}[\text{prey}|\text{sane}]\mathrm{P}[\text{sane}] + \mathrm{P}[\text{prey}|\text{crazy}]\mathrm{P}[\text{crazy}]},$$

which becomes

$$\pi_2 = \frac{\alpha\pi_1}{\alpha\pi_1 + (1-\pi_1)}. \tag{12.2}$$

Solving for α, we get (12.9). This is clearly positive, and because (12.8) is violated, it is easy to show that $\alpha < 1$. Thus we have a mixed strategy perfect Bayesian equilibrium.

12.11 Limit Pricing

1. Let $q = d(p)$ be demand for the good, $d' < 0$, and let c be constant marginal cost. Then profits are given by $\pi = d(p)(p-c)$, and the first order condition for π-maximization is

   $$\pi_p(p_m) = (p_m - c)d'(p_m) + d(p_m) = 0.$$

   We take the total derivative of this with respect to c, treating $p_m = p_m(c)$ as an implicit function of c. We get

   $$\pi_{pp}\frac{dp_m}{dc} + \pi_{pc} = 0.$$

   Now $\pi_{pp} < 0$ by the second order condition for profit maximization, and $\pi_{pc} = (\partial/\partial c)((p_m - c)d'(p_m) + d(p_m)) = -d'(p_m) > 0$. Thus $dp_m/d_c > 0$.

2. In a separating equilibrium, the low-cost firm does not want to pick the high-cost firm's price, and vice-versa. Moreover, the entrant only enters if the incumbent charges the high price. Let p^l be the price charged by the low-cost firm, and suppose that if $p^l = p^l_m$, then the high-cost firm would find it profitable to charge this price too, so we would not have a separating equilibrium. Knowing this, when $c = c_l$, the incumbent might be willing to charge $p^l < p^l_m$ (limit pricing) to make it too costly for the high-cost firm to mimic this behavior.

Thus for $c = c_h$ in a separating equilibrium, we must then have $m^h + \delta d^h \geq m^h(p^l) + \delta m^h$, where δ is the discount factor, since charging p^h_m must be better than charging p^l. Also, for $c = c_l$, we must have $m^l(p^l) + \delta m^l \geq m^l + \delta d^l$, since the low-cost firm must prefer charging p^l and being a monopolist in the second period (charging p^l_m) to charging p^h in the first period and being a duopolist in the second. Finally, limit pricing requires $p^l < p^l_m$, so at $c = c_h$, the incumbent would prefer to charge p^l_m if that convinced the entrant not to enter. Thus we must have $m^h(p^l_m) + \delta m^h > m^h + \delta d^h$. In sum, for limit pricing to obtain (i.e., a separating equilibrium with $p^l < p^l_m$) we must have the following three inequalities holding simultaneously:

$$\begin{aligned} m^h - m^h(p^l) &\geq \delta(m^h - d^h) \\ \delta(m^h - d^h) &\geq m^h - m^h(p^l_m) \\ \delta(m^l - d^l) &\geq m^l - m^l(p^l) \end{aligned} \tag{12.3}$$

3. Here is the beginning of an example. It's not complete. Suppose demand is given by $q = d(p) = 1 - p$, so monopoly profits are $m = (1-p)(p-c) - f$ where c is marginal cost and f is fixed cost. The first order condition for monopoly profits gives $p_m = (1+c)/2$, so $m(p_m) = (1-c)^2/4 - f$. Notice that, as expected, monopoly price is an increasing function of marginal cost.

   In the duopoly case, we assume Cournot duopoly. Let q_1 and q_2 be the quantities of the two firms, so $q = q_1 + q_2$, and price is $p = 1 - q = 1 - q_1 - q_2$. Duopoly profits for firm 1 are $d_1(q_1) = (1 - q_1 - q_2 - c)q_1 - f$, and the first order condition for profit maximization is $q_1 = (1 - q_2 - c)/2$. Let c_2 be the cost of the entrant, so $d_2(q_2) = (1 - q_1 - q_2 - c_2)q_2 - f$, and the first order condition gives $q_2 = (1 - q_1 - c_2)/2$. Solving simultaneously (the intersection of the offer curves), we get $q_1 = (1 + c_2 - 2c)/3, q_2 = (1 + c - 2c_2)/3$, so $p = 1 - q_1 - q_2 = (1 + c + c_2)/3$. We then have $d_1 = (1 + c_2 - 2c)^2/9 - f$, $d_2 = (1 + c - 2c_2)^2/9 - f$. The condition $d^h_2 > 0 > d^l_2$ becomes $c^h - 2c_2 > 3\sqrt{f} - 1 > c^l - 2c_2$, which will hold for some f as long as $c^h > c^l$ and $c^h > 2c_2 - 1$.

   Suppose $\delta = 1$ for simplicity. In a separating equilibrium, the l and h types do different things, and the entrant uses the signal to decide whether or not to enter. In the simplest case, an l incumbent would charge $p^l = p^l_m$ and the h incumbent would not want to follow suit.

This requires that $m^h + d^h \geq m^h(p^l_m) + m^h$, or $d^h \geq m^h(p^l_m)$, or $(1 + c_2 - 2c^h)^2/9 \geq (1 - c^l)(1 + c^l - 2c^h)/4$, or

$$c_2 \geq 3[(1 - c^l)(1 + c^l - 2c^h)]^{1/2}/2 - 1 + 2c^h.$$

For concreteness, let's take $c^l = 0.1$, $c^h = 0.3$. Then this inequality becomes $c_2 \geq 0.6062$. Note that $c^h = 0.3 > 2c_2 - 1 = 0.2124$, so the condition $c^h > 2c_2 - 1$ holds as long as $c_2 < 0.65$. We must also show that the l firm prefers choosing p^l_m over p^h_m, which is obvious, since choosing p^l_m gives higher profits in the first period and keeps the competitor out. Conclusion: we have a separating equilibrium without limit pricing when $c^l = 0.1$, $c^h = 0.3$, $0.6062 \leq c_2 < 0.65$, and f is chosen so that $c^h - 2c_2 > 3\sqrt{f} - 1 > c^l - 2c_2$ is satisfied.

But suppose $c_2 < 0.6062$, which is likely for a potentially profitable firm, since these marginal costs are twice as high as c^h. The l firm can still signal the fact that it has low costs by charging $p^l < p^l_m$, if this is a strategy that the h firm would not follow. The h incumbent will not charge p^l if $m^h + d^h > m^h(p^l) + m^h$ (i.e., charging the monopoly price in the first period and facing a duopoly in the second is better than imitating the l firm in the first and being a monopoly in the second), or $d^h > m^h(p^l)$, which is $(1 + c_2 - 2c^h)^2/9 > (1 - p^l)(p^l - c^h)$. Treating this as an equality to determine the highest possible p^l with this property, we get

$$p^{l2} - p^l(1 + c^h) + (1 + c_2^2 + 4c^{h2} + 2c_2 + 5c^h - 4c_2c^h)/9 = 0,$$

which is a quadratic equation. Using $c_2 = c^h = 0.3$, this becomes

$$p^{l2} - 1.3p^l(1 + c^h) + 0.354444 = 0,$$

whose relevant root is $p^l = 0.389125$. Notice that $p^l_m = (1+c^l)/2 = 0.55$, so this is a significant case of limit pricing. Is this the best strategy for the l firm? By doing so, it receives $m(p^l) + m^l - 2f$, since the entrant does not enter in the second period. By charging p^l, the high cost firm would do the same thing, so say the entrant enters (this depends on the entrant's subjective prior concerning the probability that the incumbent is an h or an l). Then the l firm earns $m^l + d^l - 2f$. Thus limit pricing is Nash if $m(p^l) + m^l > m^l + d^l$, or $m(p^l) > d^l$. But $m(p^l) = (p^l - c^l)(1 - p^l) = 0.176619$ and $d^l = 0.2025$, so this fails.

12.12 A Simple Limit Pricing Model

If demand is $q = m - p$ and constant marginal cost is c, it is straightforward to check that the profit-maximizing price p^m, quantity q^m, and profit π^m for a monopolist are given by

$$q^m = \frac{m-c}{2}, \qquad p^m = \frac{m+c}{2}, \qquad \pi^m = \frac{(m-c)^2}{4}$$

provided $m \geq c$. Now suppose there are two firms in a Cournot duopoly, with costs c_1 and c_2, respectively. We assume c_1 and c_2 are common knowledge. It is straightforward to check that the optimal quantities q_i^d, prices p_i^d and profits π_i^d for $i = 1, 2$ are given by

$$q_1^d = \frac{m + c_2 - 2c_1}{3}, \qquad q_2^d = \frac{m + c_1 - 2c_2}{3},$$

$$p^d = \frac{m + c_1 + c_2}{3}, \quad \pi_1^d = \frac{(m + c_2 - 2c_1)^2}{9}, \quad \pi_2^d = \frac{(m + c_1 - 2c_2)^2}{9}.$$

Also, the constraint that quantities be nonnegative requires $2c_1 \leq m + c_2$ and $2c_2 \leq m + c_1$. If one of the nonnegativity constraints is violated, it is easy to see that the higher cost firm will not be in the market, and the lower cost firm will charge the monopoly price. For instance, suppose $c_1 > (m + c_2)/2$. If firm 2 produces the monopoly quantity, then $q_2 = (m - c_2)/2$, so

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= m - 2q_1 - q_2 - c_1 = m - 2q_1 - (m - c_2)/2 - c_1 \\ &< m - 2q_1 - (m - c_2)/2 - (m + c_2)/2 = -2q_1, \end{aligned}$$

so the optimal response for firm 1 is to stay out of the market (i.e., to set $q_1 = 0$).

Suppose firm 1 is the incumbent (A) and firm 2 is the potential entrant (B), so by assumption $c_2 = c_h$. If firm 2 knew firm 1's costs, firm 2 would enter if and only if

$$\pi_2^d = (m + c_1 - 2c_h)^2/9 > f,$$

where f is the fixed entry cost. By assumption, this inequality must hold when $c_1 = c_h$ and fail when $c_1 = c_l$; i.e., we must have

$$(m + c_l - 2c_h)^2/9 \leq f \leq (m - c_h)^2/9.$$

In addition, it must not be profitable for A to pretend to be low cost if A is in fact high cost. We express this condition as follows. Let $(\pi_{mh}, \pi_{ml}, \pi_{dh}, \pi_{dl},$ be the monopoly profits of a high cost firm, the monopoly profits of a low cost firm, the duopoly profits of a high cost firm facing a high cost firm, and the duopoly profits of a low cost firm facing a high cost firm. Also, let π^* be the profits of a high cost monopolist pretending to be a low cost monopolist. For a separating equilibrium a high cost firm should not want to pretend to be low cost. Assuming no discounting, this means we must have $\pi^* + \pi_{mh} > \pi_{mh} + \pi_{dh}$, or

$$\pi^* < \pi_{dh}.$$

This reduces to the condition

$$(m + c_l - 2c_h)(m - c_l) > \frac{4}{9}(m - c_h)^2.$$

Notice that if c_l is close enough to c_h, this inequality holds.